Nonlinear instabilities relating to negative-energy modes

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(Received 8 March 1993)

The nonlinear instability of general linearly stable systems allowing linear negative-energy perturbations is investigated with the aid of a multiple-time-scale formalism. It is shown that the basic equations thus obtained imply resonance conditions and possess inherent symmetries which lead to the existence of similarity solutions of these equations. These solutions can be of an explosive type, oscillatory, or static. It is demonstrated that at least some of the oscillatory and static solutions are normally linearly unstable.

PACS number(s): 03.20.+i

I. INTRODUCTION

In 1925 Cherry [1] discussed two oscillators of positive and negative energy that are nonlinearly coupled in a special way, and presented a class of exact solutions of the nonlinear equations showing explosive instability independent of the strength of the nonlinearity and the initial amplitudes, although linearized theory predicts absolute stability. (For additional references on nonlinear instabilities see Weiland and Wilhelmsson [2] and Wilhelmsson [3]; see also Ref. [4].) Cherry's Hamiltonian is

$$H=-\tfrac{1}{2}\omega_1(p_1^2+q_1^2)+\tfrac{1}{2}\omega_2(p_2^2+q_2^2)+\frac{\alpha}{2}[2q_1p_1p_2-q_2(q_1^2-p_1^2)] \ . \quad (1)$$

The constant α measures the effect of nonlinearity. For $\alpha=0$ one has two uncoupled oscillators of frequencies $\omega_1>0$ and $\omega_2>0$ which possess negative and positive energy, respectively. If $\omega_2=2\omega_1$, one has a third-order resonance. For this case Cherry found the following exact two-parameter solution set:

$$q_1=\frac{\sqrt{2}}{\epsilon-\alpha t}\sin(\omega_1 t+\gamma) \ , \quad p_1=\frac{-\sqrt{2}}{\epsilon-\alpha t}\cos(\omega_1 t+\gamma) \ ,$$
$$q_2=\frac{-1}{\epsilon-\alpha t}\sin(2\omega_1 t+2\gamma) \ , \quad p_2=\frac{-1}{\epsilon-\alpha t}\cos(2\omega_1 t+2\gamma) \ , \quad (2)$$

where ϵ and γ are determined by the initial conditions. These solutions possess the mentioned features. Pfirsch [5] reformulated Cherry's example and generalized it to three oscillators satisfying the resonance condition $\sum_i\omega_i=0$. To this end, complex quantities given by

$$\xi_i=\frac{p_i+iq_i}{\sqrt{2}} \ , \quad \xi_i^*=\frac{p_i-iq_i}{\sqrt{2}} \quad (3)$$

were introduced. This constitutes a canonical transformation to ξ_i as the new coordinates and to $i\xi_i^*$ as the new momenta. Cherry's Hamiltonian then becomes

$$H=-\omega_1\xi_1^*\xi_1+\omega_2\xi_2^*\xi_2-\sqrt{2}\alpha(\xi_1^2\xi_2-\xi_1^{*2}\xi_2^*) \ . \quad (4)$$

This exhibits a simple structure of the nonlinear term which also allows a simple physical interpretation: in quantum theoretical language it means the simultaneous annihilation or creation of two quanta of frequency ω_1 with energy $-\hbar\omega_1$ each and of one quantum of frequency ω_2 with energy $+\hbar\omega_2$. If $\omega_2=2\omega_1$, these processes leave the energy unchanged and therefore allow the amplitudes to grow. The same holds for the first two terms in H. The growth of the amplitudes is, of course, only possible for perturbations with vanishing H. The generalization to three coupled oscillators is given by the Hamiltonian

$$H=\sum_{k=1}^{3}\omega_k\xi_k^*\xi_k+\alpha\xi_1\xi_2\xi_3+\alpha^*\xi_1^*\xi_2^*\xi_3^* \ . \quad (5)$$

With the frequencies ω_k satisfying the three-wave conservation law $\sum_k\omega_k=0$, a three-parameter solution set is

$$\xi_k=\left[\frac{i\alpha^*}{|\alpha|}\right]^{1/3}\frac{1}{\epsilon-|\alpha|t}e^{i\omega_k t+i\varphi_k} \ , \quad \sum_{i=1}^{3}\varphi_k=0 \ . \quad (6)$$

If the resonance condition is not satisfied, the system can be shown to be still explosively unstable; the initial perturbations, however, must exceed a certain threshold [5]. It is easy to generalize this further to an arbitrary number of oscillators, but the coupling terms are restricted to being Cherry-like. In quantum-mechanical language this means that the coupling terms consist of products of creation operators only and annihilation operators only. This is, however, not the usual situation, in which non-Cherry-like coupling terms occur in addition. In the present paper general coupling terms are investigated on the basis of a first-order multiple-time-scale formalism. This assumes that the linear solutions define fast time scales, whereas the nonlinear terms introduce additional slow variations of the dominant terms. The formalism eliminates then all but the fast-time-scale resonant terms from the equations for the lowest-order quantities. Since an explosive process is eventually a fast process, this treatment allows one to discuss only the initial, still slow nonlinear phase. That this might not be too strong a limitation is shown by the example of the motion of a charged particle ($e=m=1$) in a potential $-(x^2+y^2)/2-\epsilon x^3/3$ and a constant magnetic field B in the z direction (see the Appendix). For $B>2$ the linearized motion is stable, constituting a combination of gyro and drift motion. The gyro motion has positive energy and the drift motion negative energy. The nonlinear terms in the equations of motion couple these two types of motion and, in the case of resonance, i.e., the frequency of the gyro motion is twice that of the drift motion, they lead to nonlinear instability for almost all initial perturbations. This is found by solving the equations numerically. For initial conditions with x^2+y^2 and $\dot{x}^2+\dot{y}^2$ of order 1 an exponential runaway, essentially in the y direction, eventually occurs; this contradicts the long-time explosive behavior of the approximate solutions obtained by the first-order multiple-time-scale formalism, but the initial phase, lasting for times of the order $4\epsilon^{-1}$, is well described by them. Initial conditions with x^2+y^2 and $\dot{x}^2+\dot{y}^2$ of order ϵ^{-2} or larger indeed lead to explosive behavior. But such initial conditions are outside the range of validity of the first-order multiple-time-scale formalism. The basic equations resulting from the application of the first-order multiple-time-scale formalism to the above "particle-on-a-hill" problem are derived in the Appendix. They are identical with the reformulated two-oscillator equations of Cherry.

When the fast-time-scale resonance is not exact, a case which is not investigated in this paper, nonvanishing threshold amplitudes should play a role like in the Cherry case. Also not treated here is the influence of the fast time scales which will generally result in chaotic motion [6]. An illuminating discussion of this feature was presented by Kueny [7].

In the light of these considerations and limitations an investigation of general systems on the basis of the multiple-time-scale formalism appears to be of interest. In Sec. II the basic equations for general systems resulting from the application of the multiple-time-scale formalism are derived; Sec. III draws conclusions from the inherent symmetries of these basic equations; Sec. IV discusses similarity solutions relating to these symmetries; Sec. V presents a discussion of the similarity solutions; Sec. VI investigates the linear stability of static and oscillatory similarity solutions; Sec. VII presents examples with non-Cherry-like coupling terms.

II. MULTIPLE-TIME-SCALE FORMALISM FOR OBTAINING THE BASIC EQUATIONS

Let $\eta_i(t)$ be the dynamic variables of a Hamiltonian system, $i=1,\ldots$. The canonical conjugate to η_i be $P_i=i\eta_i^*$, where the asterisk denotes the complex-conjugate quantity. The dynamic systems we are interested in are then described by Hamiltonians of the form

$$H=\sum_i \omega_i\eta_i^*\eta_i+\epsilon U(\eta_k,\eta_k^*) \ , \qquad U(\eta_k,\eta_k^*)=[U(\eta_k,\eta_k^*)]^* \ . \tag{7}$$

The constants ω_i are real and can be positive or negative. The sum on the right-hand side yields linear contributions to the equations of motion, whereas ϵU leads to nonlinear terms, ϵ being a smallness parameter.

The equations of motion are

$$\dot{\eta}_i=-i\omega_i\eta_i-i\epsilon\frac{\partial U}{\partial\eta_i^*} \ . \tag{8}$$

With

$$\eta_i=e^{-i\omega_i t}\xi_i \tag{9}$$

one obtains

$$\dot{\xi}_i=-i\epsilon e^{i\omega_i t}\frac{\partial U}{\partial\eta_i^*}\bigg|_{\eta_k=e^{-i\omega_k t}\xi_k,\ \eta_k^*=e^{+i\omega_k t}\xi_k^*} \ . \tag{10}$$

For the initial phase of a nonlinear instability one can assume a slow time evolution compared with that corresponding to the ω_i's. The nonlinear time scales must relate to ϵ and therefore a multiple-time-scale formalism should be adequate to obtain an approximate solution. The perturbation treatment consists in an expansion

$$\xi_i(t)=\sum_\nu \epsilon^\nu\xi_i^{(\nu)}(\tau_0,\tau_1,\ldots) \ , \quad \tau_n=\epsilon^n t \ , \qquad \frac{d}{dt}=\sum_n \epsilon^n\frac{\partial}{\partial\tau_n} \ . \tag{11}$$

Up to first order this yields

$$\frac{\partial\xi_i^{(0)}}{\partial\tau_0}=0 \ , \qquad \frac{\partial\xi_i^{(1)}}{\partial\tau_0}+\frac{\partial\xi_i^{(0)}}{\partial\tau_1}=-ie^{i\omega_i\tau_0}\frac{\partial U}{\partial\eta_i^*}\bigg|_{\eta_k=e^{-i\omega_k\tau_0}\xi_k^{(0)},\ \eta_k^*=e^{+i\omega_k\tau_0}\xi_k^{(0)*}} \ . \tag{12}$$

From these equations one obtains

$$\xi_i^{(1)}(\tau_0,\tau_1)=\xi_i^{(1)}(0,\tau_1)+\int_0^{\tau_0}\left[-\frac{\partial\xi_i^{(0)}}{\partial\tau_1}-ie^{i\omega_i\tau_0'}\frac{\partial U}{\partial\eta_i^*}\bigg|_{\eta_k=e^{-i\omega_k\tau_0'}\xi_k^{(0)},\ \eta_k^*=e^{+i\omega_k\tau_0'}\xi_k^{(0)*}}\right]d\tau_0' \ . \tag{13}$$

The requirement that $\xi_i^{(1)}$ must stay finite for $\tau_0\to\infty$ yields an equation for $\xi_i^{(0)}$ concerning its τ_1 dependence:

$$\frac{\partial\xi_i^{(0)}}{\partial\tau_1}=-i\lim_{\tau_0\to\infty}\frac{1}{\tau_0}\int_0^{\tau_0}e^{i\omega_i\tau_0'}\frac{\partial U}{\partial\eta_i^*}\bigg|_{\eta_k=e^{-i\omega_k\tau_0'}\xi_k^{(0)},\ \eta_k^*=e^{+i\omega_k\tau_0'}\xi_k^{(0)*}}d\tau_0' \ . \tag{14}$$

When

$$W(\xi_k^{(0)},\xi_k^{(0)*}) \equiv \epsilon \lim_{\tau_0\to\infty} \frac{1}{\tau_0}\int_0^{\tau_0} U(e^{-i\omega_k\tau_0'}\xi_k^{(0)}, e^{+i\omega_k\tau_0'}\xi_k^{(0)*})d\tau_0' \tag{15}$$

is introduced and the upper index 0 of $\xi_k^{(0)}$ is dropped, Eq. (14) becomes

$$\frac{d\xi_i}{dt} = -i\frac{\partial W}{\partial \xi_i^*} . \tag{16}$$

This equation derives also from the Lagrangian

$$L = i\sum_i \xi_i^*\dot{\xi}_i - W(\xi_k,\xi_k^*) . \tag{17}$$

From now on the Lagrangian (17) will be the basis for further investigations.

III. INHERENT SYMMETRIES

We first note that corresponding to the reality property (7) of U it holds that

$$W(\xi_k^{(0)},\xi_k^{(0)*}) = [W(\xi_k^{(0)},\xi_k^{(0)*})]^* . \tag{18}$$

Let us now assume that W is a homogeneous function of degree n. From its definition (15) the following property of W then results:

$$W(e^{(\alpha+i\omega_i)\varphi}\xi_i, e^{(\alpha-i\omega_i)\varphi}\xi_i^*) = e^{n\alpha\varphi}W(\xi_i,\xi_i^*) , \tag{19}$$

where α and φ are real quantities. For $\alpha=0$ this relation also holds for general W. From Eq. (19) one derives

$$\begin{aligned}\frac{d}{d\varphi}W(e^{(\alpha+i\omega_i)\varphi}\xi_i, e^{(\alpha-i\omega_i)\varphi}\xi_i^*) &= \sum_i \left[(\alpha+i\omega_i)\xi_i\frac{\partial}{\partial \xi i} + (\alpha-i\omega_i)\xi_i^*\frac{\partial}{\partial \xi_i^*}\right] \\ &\quad\times W(e^{(\alpha+i\omega_i)\varphi}\xi_i, e^{(\alpha-i\omega_i)\varphi}\xi_i^*) \\ &= n\alpha e^{n\alpha\varphi}W(\xi_i,\xi_i^*) .\end{aligned} \tag{20}$$

When taking this relation at $\varphi=0$ its real and imaginary parts yield

$$\begin{aligned}&\sum_i \left[\xi_i\frac{\partial}{\partial\xi_i} + \xi_i^*\frac{\partial}{\partial\xi_i^*}\right]W = nW , \\ &\sum_i \omega_i\left[\xi_i\frac{\partial}{\partial\xi_i} - \xi_i^*\frac{\partial}{\partial\xi_i^*}\right]W = 0 ,\end{aligned} \tag{21}$$

where the second relation is also valid for general W. Application of the equations of motion to relations (21) yields

$$\begin{aligned}&\sum_i (\xi_i\dot{\xi}_i^* - \xi_i^*\dot{\xi}_i) = inW , \\ &\sum_i \omega_i(\xi_i\dot{\xi}_i^* + \xi_i^*\dot{\xi}_i) = 0 .\end{aligned} \tag{22}$$

The second of these equations yields the new constant of the motion for general W:

$$\sum_i \omega_i|\xi_i|^2 = \text{const} . \tag{23}$$

This is just the energy of the linearized motion. From the Lagrangian (17) it also follows that

$$W(\xi_i,\xi_i^*) = \text{const} . \tag{24}$$

Hence both the linear and nonlinear contributions to the total energy are constants of the motion.

IV. SIMILARITY SOLUTIONS

From relation (19) it follows that the Lagrangian (17) is invariant to the transformation

$$\begin{aligned}&\xi_i = e^{(\alpha+i\omega_i)\varphi}\mu_i , \\ &\xi_i^* = e^{(\alpha-i\omega_i)\varphi}\mu_i^* , \\ &t = e^{-(n-2)\alpha\varphi}\tau ,\end{aligned} \tag{25}$$

with α and φ real and time independent. This means that μ_i is a solution of

$$\frac{d}{d\tau}\mu_i = -i\frac{\partial W(\mu_i,\mu_i^*)}{\partial \mu_i^*} \tag{26}$$

if ξ_i solves Eq. (16). This allows us to construct a similarity solution $\mu_i(t)$ in the following way:

$$\xi_i(t) = e^{(\alpha+i\omega_i)\varphi}\mu_i(e^{(n-2)\alpha\varphi}t) \stackrel{!}{=} \mu_i(t+\tau_\varphi) . \tag{27}$$

The exclamation mark above the equality sign denotes that $\mu_i(t)$ is required to be a similarity solution. For φ and τ_φ infinitesimally small the required relation becomes

$$(\alpha+i\omega_i)\varphi\mu_i(t) + (n-2)\alpha\varphi t\dot{\mu}_i(t) = \tau_\varphi\dot{\mu}_i(t) . \tag{28}$$

The relation between φ and τ_φ is obtained by taking this relation at $t=0$:

$$(\alpha+i\omega_i)\varphi\mu_i(0) = \tau_\varphi\dot{\mu}_i(0) . \tag{29}$$

Since φ and τ_φ must be real and independent of the index i, Eq. (29) implies a restriction on the initial conditions of μ_i and $\dot{\mu}_i$, which can be stated as

$$\beta \equiv \frac{\varphi}{\tau_\varphi} = \frac{\dot{\mu}_i(0)}{\mu_i(0)}\frac{1}{\alpha+i\omega_i} \tag{30}$$

must be real and independent of i.

With Eq. (29) the following equation for $\mu_i(t)$ results:

$$\left[1 - \frac{(n-2)\alpha}{\alpha+i\omega_i}\frac{\dot{\mu}_i(0)}{\mu_i(0)}t\right]\dot{\mu}_i(t) = \frac{\dot{\mu}_i(0)}{\mu_i(0)}\mu_i(t) . \tag{31}$$

The general solution of this equation is

$$\mu_i(t) = \frac{\mu_i(0)}{[1-(n-2)\alpha\beta t]^{(\alpha+i\omega_i)/\alpha(n-2)}} , \tag{32}$$

where the definition (30) has been used. For $\alpha\to\infty$ and $\alpha\beta\neq 0$ the simple explosive behavior known for the Cherry oscillators is recovered. For $\alpha\to 0$ the solution (32) becomes purely oscillatory:

$$\mu_i(t)=\mu_i(0)e^{i\beta\omega_i t} . \tag{33}$$

A possible situation is also that $\beta=0$ and α finite. This means

$$\mu_i(t)=\mu_i(0) . \tag{34}$$

We conclude this section with the derivation of the equations for the $\mu_i(0)$'s, α and β. These equations are obtained from Eq. (30) on elimination of $\dot{\mu}_i(0)$ by means of the equations of motion. This transforms Eq. (30) into

$$\beta(\alpha+i\omega_i)\mu_i(0)+i\frac{\partial W(\mu_i(0),\mu_i^*(0))}{\partial\mu_i^*(0)}=0 . \tag{35}$$

α and β have the character of eigenvalues for these nonlinear equations.

V. DISCUSSION OF SIMILARITY SOLUTIONS

We start this section by noting the following transformation property of Eq. (35):

$$\mu_i(0)=e^{A+iB\omega_i}m_i(0) , \quad e^A \text{ and } B \text{ are real} , \tag{36}$$

transforms Eq. (35) into

$$e^{-(n-2)A}\beta(\alpha+i\omega_i)m_i(0)+i\frac{\partial W(m_i(0),m_i^*(0))}{\partial m_i^*(0)}=0 . \tag{37}$$

This means that also m_i is a solution of Eq. (35), but with a modified β. In order to study the effect of this transformation on the similarity solutions, we rewrite Eq. (32) as

$$\begin{aligned}\mu_i(t)&=\frac{\mu_i(0)}{[(n-2)\alpha\beta]^{(\alpha+i\omega_i)/\alpha(n-2)}}\\&\quad\times\frac{1}{(t_e-t)^{(\alpha+i\omega_i)/\alpha(n-2)}} ,\\ t_e&=\frac{1}{(n-2)\alpha\beta} .\end{aligned} \tag{38}$$

A similar solution can be written for $m_i(t)$. Expressing the quantities $m_i(0)$ and the corresponding β in this solution by the quantities $\mu_i(0)$ and the original β one obtains

$$\begin{aligned}m_i(t)&=e^{i(A/\alpha-B)\omega_i}\frac{\mu_i(0)}{[(n-2)\alpha\beta]^{(\alpha+i\omega_i)/\alpha(n-2)}}\\&\quad\times\frac{1}{(\hat{t}_e-t)^{(\alpha+i\omega_i)/\alpha(n-2)}} ,\\ \hat{t}_e&=e^{(n-2)A}t_e .\end{aligned} \tag{39}$$

The main effect of the transformation is therefore a change of the explosion time t_e, whereas α remains unchanged.

A special transformation of Eq. (35), which corresponds to $A=i\pi$, is

$$\mu_i=-m_i , \quad \beta=(-1)^n\hat{\beta} , \tag{40}$$

where $\hat{\beta}$ is the β corresponding to m_i.

We now find α and β from Eq. (35) in terms of the $\mu_i(0)$'s. From these equations one derives

$$\begin{aligned}&\sum_i\beta(\alpha+i\omega_i)|\mu_i|^2+i\sum_i\mu_i^*\frac{\partial W(\mu_i(0),\mu_i^*(0))}{\partial\mu_i^*(0)}=0 ,\\&\sum_i\beta(\alpha+i\omega_i)\omega_i|\mu_i|^2+i\sum_i\omega_i\mu_i^*\frac{\partial W(\mu_i(0),\mu_i^*(0))}{\partial\mu_i^*(0)}=0 .\end{aligned} \tag{41}$$

The imaginary part of the first and the real part of the second equation yields, by means of relations (21),

$$2\beta\sum_i\omega_i|\mu_i|^2=-nW , \quad 2\alpha\beta\sum_i\omega_i|\mu_i|^2=0 . \tag{42}$$

These relations have the important consequence that

$$\alpha W=0 . \tag{43}$$

$\alpha\neq0$ requires therefore $W=0$, in agreement with the fact that W can be constant in this case only if it vanishes.

If $\sum_i\omega_i|\mu_i|^2\neq0$ holds, then

$$\alpha\beta=0 \tag{44}$$

follows. Furthermore, $\sum_i\omega_i|\mu_i|^2=0$ implies $W=0$.

The real part of the first and the imaginary part of the second of the relations (41) yield equations for α and β which can be used under all circumstances, also when $\sum_i\omega_i|\mu_i|^2=0$, in which case Eqs. (42) do not contain α and β anymore:

$$\begin{aligned}\alpha\beta&=-i\frac{\sum_i\left[\mu_i^*\frac{\partial W}{\partial\mu_i^*}-\mu_i\frac{\partial W}{\partial\mu_i}\right]}{2\sum_i|\mu_i|^2} ,\\ \beta&=-\frac{\sum_i\omega_i\left[\mu_i^*\frac{\partial W}{\partial\mu_i^*}+\mu_i\frac{\partial W}{\partial\mu_i}\right]}{2\sum_i\omega_i^2|\mu_i|^2} .\end{aligned} \tag{45}$$

An example is the three Cherry oscillator case, for which $\mu_i^*(\partial W/\partial\mu_i^*)$ is independent of i and $\sum_i\omega_i=0$. With these properties the second of Eq. (45) yields $\beta=0$. On the other hand, the first of Eqs. (45) guarantees $\alpha\beta\neq0$. Hence, for the Cherry oscillators $\alpha=\infty$ holds and therefore the simple form of an explosive instability is recovered.

VI. STABILITY OF STATIC AND OSCILLATORY SIMILARITY SOLUTIONS

The preceding section showed that there can exist systems which possess only static or oscillatory similarity solutions but not explosive ones. The question is then whether these static or oscillatory similarity solutions are stable or not. This question is now investigated by linearizing the equations of motion around such similarity solutions $\mu_i(t)$.

The starting point is Eq. (16) together with Eq. (33). We introduce perturbations by

$$\xi_i(t)=[\mu_i(0)+\delta\xi_i(t)]e^{i\beta\omega_i t} . \tag{46}$$

The first-order contributions to Eq. (16) are

$$\delta\dot{\xi}_i+i\beta\omega_i\delta\xi_i=-i\frac{\partial^2 W}{\partial\xi_i^*\partial\xi_k}\delta\xi_k-i\frac{\partial^2 W}{\partial\xi_i^*\partial\xi_k^*}\delta\xi_k^* ,$$
$$\delta\dot{\xi}_i^*-i\beta\omega_i\delta\xi_i^*=+i\frac{\partial^2 W^*}{\partial\xi_i\partial\xi_k^*}\delta\xi_k^*+i\frac{\partial^2 W^*}{\partial\xi_i\partial\xi_k}\delta\xi_k , \tag{47}$$

where the derivatives of the function W are at $\xi_i(t)=\mu_i(t)$ and $\xi_i^*(t)=\mu_i^*(t)$. One could now split these equations into real and imaginary parts. Let this splitting for $\delta\xi_i$ and $\delta\xi_i^*$ be

$$\delta\xi_i=u_i+iv_i , \quad \delta\xi_i^*=\mu_i-iv_i . \tag{48}$$

In the resulting equations for u_i and v_i one can make the *Ansätze*

$$u_i=u_{ci}e^{\lambda t} , \quad v_i=v_{ci}e^{\lambda t} . \tag{49}$$

λ, u_{ci}, and v_{ci} can then be complex again. It is, however, not necessary to perform the splitting explicitly. Equation (49) means that

$$\delta\xi_i\propto e^{\lambda t} , \quad \delta\xi_i^*\propto e^{\lambda t} \tag{50}$$

can also be assumed, but then it no longer holds that these quantities are complex conjugate to each other. If one denotes these quantities by $\delta\hat{\xi}_i$ and $\delta\widehat{\xi^*}_i$, one obtains from them u_{ci}, v_{ci}, and $\delta\xi_i$ according to

$$\tfrac{1}{2}(\delta\hat{\xi}_i+\delta\widehat{\xi^*}_i)=u_{ci}e^{\lambda t} ,$$
$$\frac{1}{2i}(\delta\hat{\xi}_i-\delta\widehat{\xi^*}_i)=v_{ci}e^{\lambda t} , \tag{51}$$
$$\mathrm{Re}(u_{ci}e^{\lambda t})+i\,\mathrm{Re}(v_{ci}e^{\lambda t})=\delta\xi_i .$$

With the quantities $\delta\hat{\xi}_i$ and $\delta\widehat{\xi^*}_i$ Eqs. (47) become

$$(\lambda+i\beta\omega_i)\delta\hat{\xi}_i=-i\frac{\partial^2 W}{\partial\xi_i^*\partial\xi_k}\delta\hat{\xi}_k-i\frac{\partial^2 W}{\partial\xi_i^*\partial\xi_k^*}\delta\widehat{\xi^*}_k ,$$
$$(\lambda-i\beta\omega_i)\delta\widehat{\xi^*}_i=+i\frac{\partial^2 W^*}{\partial\xi_i\partial\xi_k^*}\delta\widehat{\xi^*}_k+i\frac{\partial^2 W^*}{\partial\xi_i\partial\xi_k}\delta\hat{\xi}_k . \tag{52}$$

These equations form a non-Hermitian eigenvalue problem with, in general, complex eigenvalues λ. If the real part of one eigenvalue λ is positive, the perturbation is unstable. From the transformation property (40) it follows that for odd n with λ the value $-\lambda$ also occurs as an eigenvalue for an unperturbed state which results from the original one by the transformation (40). Hence, if for odd n the real part of λ does not vanish, an unstable perturbation exists. Because of the non-Hermitian character of the eigenvalue problem this should be the normal situation. This is illustrated in the next section by means of examples. These examples show also that the $\delta\xi_i$'s can sometimes be nonlinearly limited to the order of the μ_i's.

VII. EXAMPLES WITH NON-CHERRY-LIKE COUPLING TERMS

A. Three-mode example

The simplest example involving negative- and positive-energy modes is given by

$$W=\gamma\xi_{-1}\xi_2^*\xi_3+\gamma^*\xi_{-1}^*\xi_2\xi_3^* ,$$
$$\omega_{-1}=-1 , \quad \omega_2=2 , \quad \omega_3=3 . \tag{53}$$

The corresponding equations of motion together with Eq. (35) are

$$\dot{\xi}_{-1}=-i\gamma^*\xi_2\xi_3^*=[\beta(\alpha-i)\mu_{-1}] , \tag{54}$$

$$\dot{\xi}_2=-i\gamma\xi_{-1}\xi_3=[\beta(\alpha+2i)\mu_2] , \tag{55}$$

$$\dot{\xi}_3=-i\gamma^*\xi_{-1}^*\xi_2=[\beta(\alpha+3i)\mu_3] . \tag{56}$$

These equations have similarity solutions $\mu_i(t)$ corresponding to $\beta=0$ with

$$\mu_{-1}\neq 0 , \quad \mu_2=\mu_3=0 , \tag{57a}$$

$$\mu_2\neq 0 , \quad \mu_{-1}=\mu_3=0 , \tag{57b}$$

$$\mu_3\neq 0 , \quad \mu_{-1}=\mu_2=0 . \tag{57c}$$

The corresponding nontrivial linearized equations are

$$\delta\dot{\xi}_2=-i\gamma\mu_{-1}\delta\xi_3 , \quad \delta\dot{\xi}_3=-i\gamma^*\mu_{-1}^*\delta\xi_2 , \tag{58a}$$

$$\delta\dot{\xi}_{-1}=-i\gamma^*\mu_2\delta\xi_3^* , \quad \delta\dot{\xi}_3=-i\gamma^*\mu_2\delta\xi_{-1}^* , \tag{58b}$$

$$\delta\dot{\xi}_{-1}=-i\gamma^*\mu_3^*\delta\xi_2 , \quad \delta\dot{\xi}_2=-i\gamma\mu_3\delta\xi_{-1} . \tag{58c}$$

From these equations one obtains

$$\delta\ddot{\xi}_2=-|\gamma|^2|\mu_{-1}|^2\delta\xi_2 , \tag{59a}$$

$$\delta\ddot{\xi}_{-1}=+|\gamma|^2|\mu_2|^2\delta\xi_{-1} , \tag{59b}$$

$$\delta\ddot{\xi}_2=-|\gamma|^2|\mu_2|^2\delta\xi_2 . \tag{59c}$$

Equations (57a), (58a), (59a), (57c), 58(c), and (59c) possess only oscillatory solutions, whereas Eqs. (57c), (58c), and (59c) also have an unstable solution. This agrees with the general expectations of the preceding section; the eigenvalues λ discussed there are, for the present example, either real or purely imaginary.

B. Four-mode example

We consider again a third-order W:

$$W=\sum_{n_1}\sum_{n_2}\sum_{n_3}\gamma_{n_1n_2n_3}\xi_{n_1}\xi_{n_2}\xi_{n_3}+\gamma^*_{n_1n_2n_3}\xi^*_{n_1}\xi^*_{n_2}\xi^*_{n_3}$$
$$+\rho_{n_1,n_2n_3}\xi^*_{n_1}\xi_{n_2}\xi_{n_3}+\rho^*_{n_1,n_2n_3}\xi_{n_1}\xi^*_{n_2}\xi^*_{n_3} . \tag{60}$$

The resonance conditions are

$$\gamma_{n_1n_2n_3}\propto\delta_{\omega_{n_1}+\omega_{n_2}+\omega_{n_3}} ,$$
$$\rho_{n_1,n_2n_3}\propto\delta_{\omega_{n_1}-\omega_{n_2}-\omega_{n_3}} , \tag{61}$$

and the equations of motion

$$i\dot{\xi}_{n_1}=\sum_{n_2}\sum_{n_3}\rho_{n_1,n_2n_3}\xi_{n_2}\xi_{n_3}+2\rho^*_{n_2,n_1n_3}\xi_{n_2}\xi^*_{n_3}+3\gamma^*_{n_1n_2n_3}\xi^*_{n_2}\xi^*_{n_3}\ . \tag{62}$$

A simple four-oscillator example is given by

$$\omega_n=n\ ,\quad n=-3,1,2,3\ . \tag{63}$$

The only nonvanishing coefficients are

$$\rho_{2,11}\equiv\rho_2\ ,\quad \rho_{3,12}\equiv\rho_3\ ,\quad \gamma_{-312}\equiv\gamma\ , \tag{64}$$

and in addition all the coefficients corresponding to their symmetry. The function W is then

$$W=6\gamma\xi_{-3}\xi_1\xi_2+6\gamma^*\xi^*_{-3}\xi^*_1\xi^*_2+\rho_2\xi^*_2\xi_1^2+\rho^*_2\xi_2\xi_1^{*2}+2\rho_3\xi^*_3\xi_1\xi_2+2\rho^*_3\xi_3\xi^*_1\xi^*_2\ , \tag{65}$$

and the equations of motion become

$$\begin{aligned} i\dot{\xi}_{-3}&=6\gamma^*\xi^*_1\xi^*_2\ ,\\ i\dot{\xi}_1&=2\rho^*_2\xi_2\xi^*_1+2\rho^*_3\xi_3\xi^*_2+6\gamma^*\xi^*_2\xi^*_{-3}\ ,\\ i\dot{\xi}_2&=\rho_2\xi_1^2+2\rho^*_3\xi_3\xi^*_1+6\gamma^*\xi^*_1\xi^*_{-3}\ ,\\ i\dot{\xi}_3&=2\rho_3\xi_1\xi_2\ . \end{aligned} \tag{66}$$

From the first and the last of these equations the following constant of the motion is obtained:

$$\xi_{-3}+\frac{3\gamma^*}{\rho^*_3}\xi^*_3=c_{-3}=\text{const}\ . \tag{67}$$

This constant of the motion has an important consequence. For an explosive instability to exist

$$\sum_i\omega_i|\xi_i|^2=0$$

must hold, as follows from Eq. (42). Also the constant of the motion (67) must vanish. The combination of these two requirements leads to

$$-3\left[\frac{9|\gamma|^2}{|\rho_3|^2}-1\right]|\xi_3|^2+|\xi_1|^2+2|\xi_2|^2=0\ . \tag{68}$$

This means that explosive instabilities are possible only for

$$9|\gamma|^2>|\rho_3|^2\ . \tag{69}$$

One can indeed prove the existence of solutions of the form

$$\xi_n(t)=\mu_n(t)=\frac{a_n}{t} \tag{70}$$

if the inequality (69) is satisfied.

If this is not the case, one can look for static and oscillatory similarity solutions and investigate their linear stability properties. A simple static similarity solution μ_i is obviously

$$\mu_1=\mu_2=0\ \text{ and }\mu_3\ ,\quad \mu_{-3}\text{ arbitrary}\ . \tag{71}$$

Linearization of Eqs. (66) around this solution with the perturbations $\propto\exp(\lambda t)$ and use of Eq. (67) yields the following set of equations:

$$\begin{aligned} &i\lambda\delta\xi_{-3}=0\ ,\\ &i\lambda\delta\xi_1=\frac{2}{\rho_3}[(|\rho_3|^2-9|\gamma|^2)\mu_3+3\gamma^*\rho_3c^*_{-3}]\delta\xi^*_2\ ,\\ &i\lambda\delta\xi_2=\frac{2}{\rho_3}[(|\rho_3|^2-9|\gamma|^2)\mu_3+3\gamma^*\rho_3c^*_{-3}]\delta\xi^*_1\ ,\\ &i\lambda\delta\xi_3=0\ . \end{aligned} \tag{72}$$

The combination of the $\delta\xi_1$ equation with the $\delta\xi^*_2$ equation leads to the dispersion relation

$$\lambda^2=\left|\frac{2}{\rho_3}[(|\rho_3|^2-9|\gamma|^2)\xi_3+3\gamma^*\rho_3c^*_{-3}]\right|^2\ , \tag{73}$$

which means instability.

Other examples are oscillatory solutions of the kind (33). Such solutions were found by Weitzner [8], who also proved that they are linearly unstable.

We conclude this section with the observation that the $\delta\xi_n$'s are nonlinearly limited to the order of the μ_n's by $\sum_i\omega_i|\xi_i|^2=$const, if condition (69) is not satisfied.

VIII. SUMMARY

A multiple-time-scale formalism is applied to investigate the nonlinear instability of linearly stable systems. Systems possessing linear negative-energy perturbations are shown to be generally nonlinearly unstable if these perturbations are in resonance with positive energy perturbations. The instabilities can be explosive instabilities or linear instabilities of nonlinear oscillatory or static solutions of the basic equations resulting from the application of the multiple-time-scale formalism. The amplitudes of these linear instabilities can sometimes be strongly limited by the fact that the energy of the linearized theory and the total nonlinear energy are constants of the motion and that there might exist at least one additional constant tying together negative- and positive-energy modes.

ACKNOWLEDGMENTS

The author would like to thank H. Weitzner for very interesting discussions and F. Pohl for his contribution to the investigations of the particle-on-a-hill problem, especially for carrying out the numerical computations.

APPENDIX: "PARTICLE ON A HILL"

The equations of motion are

$$\ddot{x}-\dot{y}B-x-4\epsilon x^2=0\ ,\quad \ddot{y}+\dot{x}B-y=0\ .$$

With

$$x+iy=\left[\frac{B^2}{4}-1\right]^{-1/4}\eta$$

the linearized equations become

$$\ddot{\eta}+iB\dot{\eta}-\eta=0\ .$$

The *Ansatz*

$$\eta\propto e^{-i\omega t}$$

leads to the dispersion relation

$$-\omega^2+\omega B-1=0$$

with the solutions

$$\omega_+=\frac{B}{2}+\left[\frac{B^2}{4}-1\right]^{1/2}\equiv\omega_1\ ,$$

$$\omega_-=\frac{B}{2}-\left[\frac{B^2}{4}-1\right]^{1/2}\equiv-\omega_2\ .$$

These frequencies have the properties

$$\omega_++\omega_-=\omega_1-\omega_2=B\ ,$$

$$\omega_+-\omega_-=\omega_1+\omega_2=2\left[\frac{B^2}{4}-1\right]^{1/2}\ ,$$

$$\omega_+\omega_-=-\omega_1\omega_2=1\ ,$$

$$\omega_+^2-1=\omega_1^2-1=2\left[\frac{B^2}{4}-1\right]^{1/2}\omega_+$$

$$=2\left[\frac{B^2}{4}-1\right]^{1/2}\omega_1\ ,$$

$$\omega_-^2-1=\omega_2^2-1=-2\left[\frac{B^2}{4}-1\right]^{1/2}\omega_-$$

$$=2\left[\frac{B^2}{4}-1\right]^{1/2}\omega_2\ .$$

We can now represent

$$x+iy=\eta_1+\eta_2^*$$

with the Lagrangian for η_1 and η_2 corresponding to the nonlinear equations of motion given by

$$L=i\eta_1^*(\dot\eta_1+i\omega_1\eta_1)+i\eta_2^*(\dot\eta_2+i\omega_2\eta_2)+\frac{\epsilon}{24}\left[\frac{B^2}{4}-1\right]^{-3/4}(\eta_1+\eta_1^*+\eta_2+\eta_2^*)^3\ .$$

Hence $i\eta_1^*$ and $i\eta_2^*$ are the canonical momenta to η_1 and η_2. The Hamiltonian resulting from this Lagrangian,

$$H=\omega_1|\eta_1|^2+\omega_2|\eta_2|^2-\frac{\epsilon}{24}\left[\frac{B^2}{4}-1\right]^{-3/4}(\eta_1+\eta_1^*+\eta_2+\eta_2^*)^3\ ,$$

belongs to the class of Hamiltonians defined by Eq. (7). Application of the procedure of Sec. I especially of Eq. (15), yields for $\omega_1+2\omega_2=0$ the fast time-averaged interaction Hamiltonian

$$W(\xi_1,\xi_1^*,\xi_2,\xi_2^*)=\frac{-\epsilon}{8\left[\frac{B^2}{4}-1\right]^{3/4}}[\xi_1^*\xi_2^{*2}+\xi_1\xi_2^2]\ .$$

This agrees exactly with the reformulated nonlinear interaction term of Cherry.

Figures 1–3 present a comparison of exact numerical

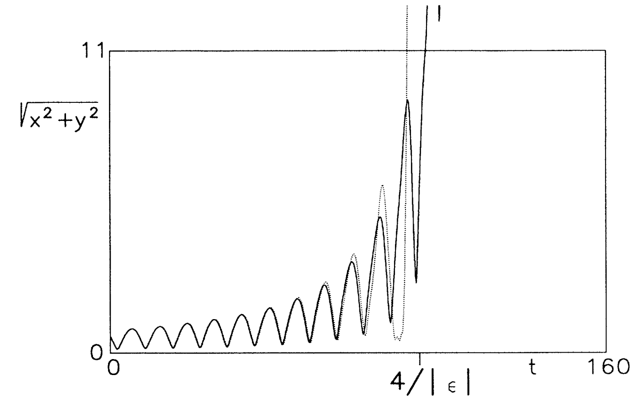

FIG. 1. Particle on a hill: $\sqrt{x^2+y^2}$ vs t for $\epsilon=0.04$, $x_0=-0.5012$, $y_0=0.3534$, $\dot x_0=0.4952$, and $\dot y_0=0.3585$. Solid line, numerical solution; dotted line, approximation.

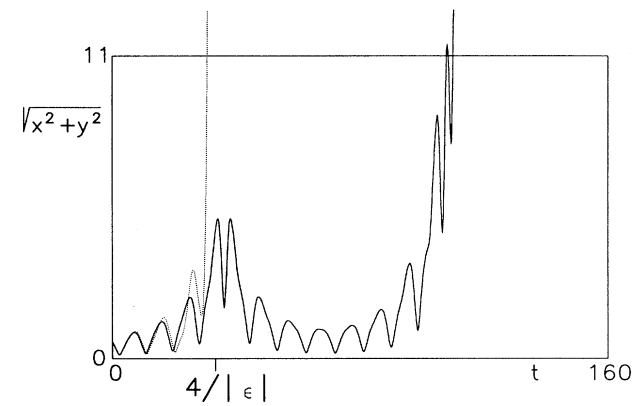

FIG. 2. Particle on a hill: $\sqrt{x^2+y^2}$ vs t for $\epsilon=0.12$, $x_0=-0.5012$, $y_0=0.3534$, $\dot x_0=0.4852$, and $\dot y_0=0.3656$. Solid line, numerical solution; dotted line, approximation.

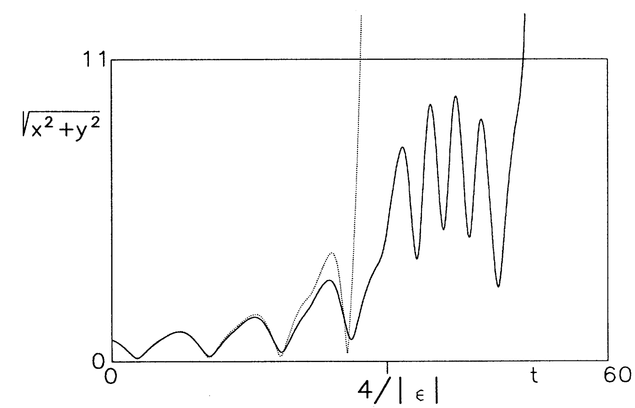

FIG. 3. Particle on a hill: $\sqrt{x^2+y^2}$ vs t for $\epsilon=0.12$, $x_0=-0.7020$, $y_0=-0.3705$, $\dot x_0=-0.2910$, and $\dot y_0=0.7354$. Solid line: numerical solution, dotted line: approximation.

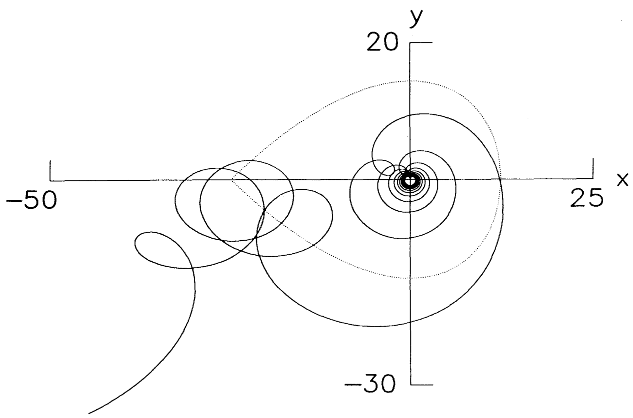

FIG. 4. Particle on a hill: Particle orbit in x,y plane together with contour line of the potential through the saddle point. The initial conditions are the same as in Fig. 1.

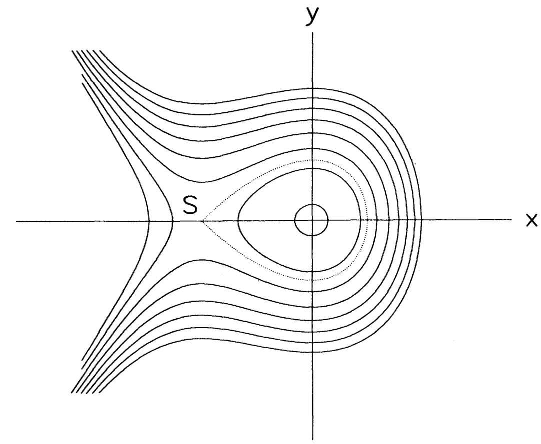

FIG. 5. Particle on a hill: Contour line plot of the potential $-(x^2+y^2)/2-\epsilon x^3/3$. *S*: saddle point.

solutions (solid lines) with approximate solutions (dotted lines) obtained by the application of the multiple-time-scale formalism up to the first order, which means inclusion of the first-order terms according to Eq. (13). All cases show good agreement between the exact and approximate solutions for the initial phase, which lasts for a time of the order $4\epsilon^{-1}$ for initial values of x^2+y^2 of the order 1. For larger times the exact and approximate solutions differ. The final runaway as described by the exact solutions sometimes occurs only after a second "attack," Fig. 2, or after dwelling for a while in the saddle-point region formed by the combination of the second- and third-order constituents of the potential, Fig. 3. This can be seen more directly in Fig. 4, which also contains a contour line of the potential through the saddle point. A full contour line picture of the potential is presented in Fig. 5. A final exponential growth, often occurring instead of an explosive behavior, is caused by the x^3 potential stopping the drift at x values with $|x| \ll |y|$ for large $|x|$; the particle then falls down the $-y^2$ potential, leading to the exponential time dependence.

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